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Comparison of General Relativity and Brans-Dicke Theory using Gravitomagnetic clock effect

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ABSTRACT: We discuss the gravitomagnetic clock effect in the context of general relativity and Brans-Dicke theory. General relativity predicts that two freely counter revolving test particles in the exterior field of a central rotating mass take different periods of time to complete the same full orbit. This phenomenon was first put forward by Cohen and Mashhoon in 1993.The gravitomagnetic clock effect also arises in the orbit of a particle which is moving round a Brans-Dicke source of mass M on the equatorial plane, and this clock effect is the same as general relativity. So, the Brans-Dicke theory cannot be distinguished from GR using GCE.

Keywords: Brans-Dicke theory, general relativity, Gravitomagnetic clock effect

I. INTRODUCTION

The gravitomagnetic clock effect (GCE) consists in the loss of synchrony of identical clocks carried around a rotating body of mass M on the equatorial plane. This effect is a consequence of general relativity and its presence has been foreseen in connection with the so called gravitomagnetism, i.e. that part of the gravitational field which, in weak field approximation behaves as the magnetic part of the electromagnetic interaction[1]. The GCE has been considered as an interesting and promising means to test the general relativistic influence of the angular momentum of a mass on the structure of space time nearby and in particular on the pace of clocks orbiting around the body[2, 3].
Actually the GCE is strictly akin to the Sagnac effect, which is a special relativistic effect induced by pure rotations, first considered as a purely classical effect by G. Sagnac[4], further on recognized in its real nature and studied by several author (see for instance [5, 6]).

The real possibility that gravitomagnetic effects can be measured with the current technology of laser ranged satellites (LAGEOS and LAGEOS II) has aroused great interest in the subject[7] . It should be mentioned that the Relativity Gyroscope Experiment (Gravity Probe B) at Stanford University, in collaboration with NASA and Lockheed-Martin Corporation, has a program of developing a space mission to detect gravitomagnetism effects directly. Certainly, these experimental programs will open new possibilities of testing general relativity against other metric theories of gravity[8,9], in particular the scalar-tensor theory, one of the most popular alternatives to Einstein theory of gravitation.
GCE is a novel effect resulting from Einstein's theory of gravitation. According to general relativity, two freely counter revolving test particles in the exterior field of a central rotating mass take different periods of time to complete the same full orbit; this time difference leads to the gravitomagnetic clock effect; first indicated by Cohen and Mashhoon in 1993[1]. The general relativistic calculation of the gravitomagnetic clock effect for a general orbit is quite complicated. The first derivation in [1] was done for equatorial circular orbits; spherical orbits have been considered in[10]; arbitrary elliptical orbits have been considered in [11]. Various theorical aspects of this effect have been investigated[12 − 17]. On the observational side, the possibility of its detection has been considered by a number of authors[18 − 23].
An addition to the scenario of GCE came in the orbit of a particle which is moving round a Brans−Dicke source of mass M on the equatorial plane in that it also contributes to the gravitomagnetic clock effect[24]; the magnitude of this contribution has been calculated in [24] and the result was confirmed and elaborated by Bini et al. [25].
This paper is organized as follows. In section 2, we give gravitomagnetic clock effect in general relativity. This was first put forward by Cohen and Mashhoon in 1993[1]. In section 3, we have examined gravitomagnetic clock effect in Brans-Dicke theory. Section 4 is devoted to some remarks.

II. GRAVITOMAGNETIC CLOCK EFFECT IN GENERAL RELATIVITY

Imagine the motion of a free non-spinning test particle on a time-like circular geodesic orbit in the equatorial plane of a Kerr black hole. The space-time metric in the Boyer Lindquist coordinates is given by

$$\mathrm{d}s^2 = -\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}t^2 + \Sigma d\theta^2 + \frac{A}{\Sigma}sin^2\theta d\varphi^2 - \frac{4Ma}{\Sigma}rsin^2\,\theta dtd\varphi, \qquad (1)$$

where

$$\Sigma = r^2 + a^2cos^2\theta, \qquad \Delta = \mathrm{r}^2 - 2M\mathrm{r} + \mathrm{a}^2, \mathrm{A} = (r^2+a^2)^2 - a^2\Delta sin^2\theta. (2)$$

Units are chosen such that G=c=1, unless specified otherwise. Here M is the mass and a=J / M is the specific angular momentum of the black hole. The geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \tau^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, (3)$$

for the radial coordinate r in the special case under consideration reduces to

$$\left(a^2 - \frac{r^3}{M}\right) d\varphi^2 - 2adtd\varphi + dt^2 = 0, (4)$$

which can be written as

$$\frac{dt}{d\varphi} = a \pm \omega_K^{-1}. (5)$$

Here, $\omega_\kappa = \sqrt{M/r^3}$, is the Keplerian angular frequency. In Eq.$(5) +$ $(-)$represent the motion of the test particle moving in the same (opposite) sense as the rotation of the black hole. To field the gravitomagnetic clock effect one integrates Eq.(5)over 2π for co-rotating and -2π for counter-rotating test particles. This results in $t_\pm = \mathrm{T}_K \pm 2\pi$a, where $\mathrm{T_K}=2\pi/\omega_\mathrm{K}$, the Keplerian period.Thus

$$t_+ - t_- = 4\pi a = \frac{4\pi J}{Mc^2} \ . \qquad (6)$$

This is the gravitomagnetic clock effect for test particle orbit, where time period of the co-rotating orbits are measured by asymptotically static inertial observers.

III. GRAVITOMAGNETIC CLOCK EFFECT IN BRANS-DICKE THEORY

The gravitomagnetic clock effect arises in orbit of a particle which is movinground a Brans-Dicke source of mass M on the equatorial plane,

$$ds^2 = [1 - \varepsilon G_0]\left[-\left(1 - \frac{2G_0 M}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2G_0 M}{rc^2}\right)\{dr^2 + r^2(d\theta^2 + sin^2\theta\, d\varphi^2)\} - \frac{4G_0 J}{rc^3} sin^2\theta\, d\varphi\, c\, dt\right]. \qquad (7)$$

The Brans-Dicke metric for the equatorial plane $\theta = \frac{\pi}{2},\ d\theta = 0$ is given by

$$ds^2 = [1 - \varepsilon G_0]$$

$$\left[-\left(1 - \frac{2G_0 M}{rc^2}\right) c^2 dt^2 + \left(1 + \frac{2G_0 M}{rc^2}\right)\{dr^2 + r^2(0 + d\varphi^2)\} - \frac{4G_0 J}{rc^3} d\varphi\, c\, dt\right], \quad (8)$$

where

$G_0 = \phi_0^{-1} = \left(\frac{2\omega+3}{2\omega+4}\right) G$and the function $\varepsilon(x)$ is a solution of the scalar fieldequation$\varepsilon = \frac{8\pi \mathrm{T}}{c^4(2\omega+3)}$, and units are such that G=c=1.

Throughout the chapter we shall use this convention unless otherwise specified.

Now from equation (7) ,

$$ds^2 = [1 - \varepsilon G_0]\left[-\left(1 - \frac{2G_0 M}{r}\right) dt^2 + \left(1 + \frac{2G_0 M}{r}\right)(dr^2 + r^2 d\varphi^2) - \frac{4G_0 J}{r} d\varphi\ dt\right]$$

Or,

$$ds^2 = [1 - \varepsilon G_0]\left[-\left(1 - \frac{2G_0 M}{r}\right) dt^2 + dr^2 + r^2 d\varphi^2 + \frac{2G_0 M}{r} dr^2 + \frac{2G_0 M}{r} r^2 d\varphi^2 - \frac{4G_0 J}{r} d\varphi\ dt\right]$$

Or,

$$ds^2 = [1 - \varepsilon G_0]$$

$$\left[-\left(1 - \frac{2G_0 M}{r}\right) dt^2 - \frac{4G_0 J}{r} d\varphi\ dt + (2G_0 M\, r + r^2) d\varphi^2 + \left(\frac{2G_0 M}{r} + 1\right) r^2\right]. \quad (9)$$

The motion of test particle of mass m in the Brans-Dicke field is governed by the geodesic equation:

$$\frac{d^2 x^\omega}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0. \qquad (10)$$

To evaluate the gravitomagnetic clock effect, we need only the geodesic equation for the radial co-ordinate r:

$$\frac{d^2 r}{ds^2} + \Gamma^r_{tt}\left(\frac{dt}{ds}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{ds}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{ds}\right)^2 + \Gamma^r_{r\varphi}\frac{dt}{ds}\frac{d\varphi}{ds} + \Gamma^r_{\varphi t}\frac{d\varphi}{ds}\frac{dt}{ds} = 0. \qquad (11)$$

Now,

$$\Gamma^r_{tt} = \frac{1}{2} g^{rm}\left(\frac{\partial g_{mt}}{\partial t} + \frac{\partial g_{mt}}{\partial t} - \frac{\partial g_{tt}}{\partial x^m}\right)$$

$$= g^{rr}\frac{\partial g_{rt}}{\partial t} - \frac{1}{2} g^{rr}\frac{\partial g_{tt}}{\partial t} = -\frac{1}{2} g^{rr} g_{tt,r}\ , \qquad (12)$$

Since$g_{rt} = 0$.

Similarly,$\Gamma^r_{rr} = \frac{1}{2} g^{rr} g_{rr,r},\ \Gamma^r_{\varphi\varphi} = -\frac{1}{2} g^{rr} g_{\varphi\varphi,r}$,

$\Gamma^r_{t\varphi} = -\frac{1}{2} g^{rr} g_{t\varphi,r}$, $\Gamma^r_{\varphi t} = -\frac{1}{2} g^{rr} g_{t\varphi,r}$. (13)

Substituting these expressions in Eq.(11) and cancelling a common factor, we get

$$-\frac{2}{g^{rr}}\frac{d^2 r}{ds^2} + g_{tt,r}\left(\frac{dt}{ds}\right)^2 - g_{rr,r}\left(\frac{dr}{ds}\right)^2 + g_{\varphi\varphi,r}\left(\frac{d\varphi}{ds}\right)^2 + 2g_{t\varphi,r}\frac{dt}{ds}\frac{d\varphi}{ds} = 0. \qquad (14)$$

Cancelling ds^2 in the denominators of the terms in Eq.(14) and dividing by $d\varphi^2$ all through in (14), we obtain

$$-\frac{2}{g^{rr}}\frac{d^2 r}{d\varphi^2} + g_{tt,r}\left(\frac{dt}{d\varphi}\right)^2 - g_{rr,r}\left(\frac{dr}{d\varphi}\right)^2 + 2g_{t\varphi,r}\frac{dt}{d\varphi} + g_{\varphi\varphi,r} = 0. \qquad (15)$$

For circular orbit r is not a function ofφ, and hence, we obtain from (15) the following radial geodesic equation in Brans-Dicke field:

$$g_{tt,r}\left(\frac{dt}{d\varphi}\right)^2 + 2g_{t\varphi,r}\frac{dt}{d\varphi} + g_{\varphi\varphi,r} = 0. \qquad (16)$$

Solution to this equation

$$\frac{dt}{d\varphi}=\frac{-2g_{t\varphi,r}\pm\sqrt{(2g_{t\varphi,r})^2-4g_{tt,r}g_{\varphi\varphi,r}}}{2g_{tt,r}}=\frac{-g_{t\varphi,r}}{g_{tt,r}}\pm\sqrt{\left(\frac{g_{t\varphi,r}}{g_{tt,r}}\right)^2-\frac{g_{\varphi\varphi,r}}{g_{tt,r}}} \quad . \tag{17}$$

Now,

$$2g_{t\varphi,r}=\frac{d}{dr}\left\{\left(-\frac{4G_0J}{r}\right)[1-\varepsilon G_0]\right\}$$

$$=\left(\frac{4G_0J}{r^2}\right)[1-\varepsilon G_0]$$

$$g_{t\varphi,r}=\left(\frac{2G_0J}{r^2}\right)[1-\varepsilon G_0]\ , \tag{18}$$

$$g_{tt,r}=\frac{d}{dr}\left\{\left(\frac{2G_0M}{r}\right)[1-\varepsilon G_0]\right\}$$

$$=-\left(\frac{2G_0M}{r^2}\right)[1-\varepsilon G_0]\ , \tag{19}$$

$$g_{\varphi\varphi,r}=\frac{d}{dr}\{(2G_0Mr+r^2)[1-\varepsilon G_0]\}$$
$$=(2G_0M+2r)[1-\varepsilon G_0]$$

$$=2(G_0M+r)[1-\varepsilon G_0]\ . \tag{20}$$

Substitution of these results into(17), we obtain

$$\frac{dt}{d\varphi}=\frac{-\left(\frac{2G_0J}{r^2}\right)[1-\varepsilon G_0]}{-\left(\frac{2G_0M}{r^2}\right)[1-\varepsilon G_0]}\pm\sqrt{\left(\frac{\left(\frac{2G_0J}{r^2}\right)[1-\varepsilon G_0]}{-\left(\frac{2G_0M}{r^2}\right)[1-\varepsilon G_0]}\right)^2-\left(\frac{2(G_0M+r)[1-\varepsilon G_0]}{-\left(\frac{2G_0M}{r^2}\right)[1-\varepsilon G_0]}\right)}$$
$$=\frac{J}{M}\pm\sqrt{\left(\frac{J}{M}\right)^2+\left(\frac{G_0Mr^2+r^3}{G_0M}\right)}$$

$$=a\pm\sqrt{a^2+\left(\frac{G_0Mr^2+r^3}{G_0M}\right)}=a\pm\sqrt{a^2+r^2+\frac{r^3}{G_0M}} \quad . \tag{21}$$

In true units this formula reduces to

$$\frac{dt}{d\varphi}=\frac{a}{c}\pm\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}} \quad . \tag{22}$$

The gravitomagnetic clock effect arises the difference in orbital period in prograde and retrograde orbits. Prograde orbit is one where the sense of orbital motion of the test body is same as the sense of rotation of the central body. In retrograde orbit the situation is opposite. Note that in Eq. (22) the $+(-)$ sign refers to prograde (retrograde) motion. The orbital period in prograde motion can be found by integrating Eq. (22) with the plus sign from 0 to 2π . The orbital period in retrograde orbit can be found by integrating Eq.(22) with the minus sign from 0 to -2π.

We indicate by t_+ the prograde period which is

$$t_+=\int_0^{2\pi}\left(\frac{a}{c}+\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}}\right)d\varphi$$

$$=2\pi\frac{a}{c}+2\pi\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}} \quad . \tag{23}$$

Similarly ,

$$t_-=\int_0^{-2\pi}\left(\frac{a}{c}-\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}}\right)d\varphi$$

$$=-2\pi\frac{a}{c}+2\pi\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}}\ , \tag{24}$$

which is the retrograde orbital period .The period difference is

$$t_+-t_-=2\pi\frac{a}{c}+2\pi\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}}+2\pi\frac{a}{c}-2\pi\sqrt{\frac{a^2}{c^2}+\frac{r^2}{c^2}+\frac{r^3}{G_0Mc^2}}$$

$$=4\pi\frac{a}{c}\ . \tag{25}$$

This is the expression for gravitomagnetic clock effect. Clearly, the retrograde orbit is faster than the prograde orbit. Hence, the Brans-Dicke theory predicts the same GCE as the general relativity does. So, the Brans-Dicke theory cannot be distinguished from GR using the GCE.

IV. FINAL REMARKS

In this paper, we have examined gravitomagnetic clock effect in the context of general relativity and Brans-Dicke theory. In section 2, the simplest case of gravitomagnetic clock effect has been shown in general relativity where we have followed the first derivation of the GCE by Cohen and Mashhon[1]. In section 3, we have examined gravitomagnetic clock effect that

arises in the orbit of a particle which is moving round a Brans-Dicke source of mass M on the equatorial plane, and hence this clock effect is the same as general relativity. So, the Brans−Dicke theory cannot be distinguished from GR using the GCE.

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